Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

The Binomial Equation $x^p - 1 = 0$.

By Charlotte Angas Scott, *Bryn Mawr College.*

Quartisection and Quinquisection.

The equation for Quartisection may be written in a form somewhat simpler than that given by Professor Cayley in the Proceedings of the London Mathematical Society, Vol. XI (1879), pp. 11–14, viz.:

$$\begin{aligned}&\eta^4 + \eta^3 - \eta^2 \{\tfrac{1}{4}(p-1) + l + m\} \\ &\quad + \eta \{\tfrac{1}{4}(f+h)(p-1) - \tfrac{1}{2}k(p+1) + 2(g-k)(l+m)\} \\ &\quad + lm - \tfrac{1}{4}(l-m)^2(p-1) = 0,\end{aligned}$$

where
$$\begin{aligned}X^2 &= (a, b, c, d)(X, Y, Z, W), \\ XY &= (f, g, h, k)(X, Y, Z, W), \\ XZ &= (l, m, l, m)(X, Y, Z, W),\end{aligned}$$

and
$$l + m = \tfrac{1}{8}(p-1) \text{ or } \tfrac{1}{8}(3p+1)$$

according as
$$p \equiv 1 \text{ or } 5 \pmod{8},$$

$$f + g + h + k = \tfrac{1}{4}(p-1), \tag{1}$$
$$a + b + c + d = -1 - 2(f+g+h+k) - 2(l+m). \tag{2}$$

The simplification is in the coefficient of η, which may be written

$$\tfrac{1}{2}\{p(l-m) - (l+m)\};$$

and then we have for the equation for Quartisection

$$\begin{aligned}&\eta^4 + \eta^3 - \{\tfrac{1}{4}(p-1) + l + m\}\eta^2 \\ &\quad + \tfrac{1}{2}\{p(l-m) - (l+m)\}\eta \\ &\quad - \tfrac{1}{4}\{p(l-m)^2 - (l+m)^2\} = 0,\end{aligned}$$

where $l - m$ is the only quantity whose value in terms of p is unknown.

To find the coefficient of η, viz. $-\Sigma XYZ$,

$$\begin{aligned}
\Sigma XYZ &= YZW+ZWX+WXY+XYZ \\
&= Z.YW+W.ZX+X.WY+Y.XZ \\
&= (Y+W)XZ+(X+Z)YW \\
&= (Y+W)(l,\,m,\,l,\,m)+(X+Z)(m,\,l,\,m,\,l) \\
&= 2l(YZ+ZW+XY+WX) \\
&\quad +m(X^2+Y^2+Z^2+W^2+2XZ+2YW) \\
&= -2l(f+g+h+k) \\
&\quad -m(a+b+c+d+2l+2m) \\
&= -2l(f+g+h+k) && \text{[by (2)} \\
&\quad +m\{1+2(f+g+h+k)\} \\
&= -(l-m)\frac{p-1}{2}+m && \text{[by (1)} \\
&= -\tfrac{1}{2}\{p(l-m)-(l+m)\},
\end{aligned}$$

$\therefore$ coefficient of $\eta=\frac{1}{2}\{p(l-m)-(l+m)\}$.

In the Proc. Lond. Math. Soc., Vol. XII, pp. 15–16, and Vol. XVI, pp. 61–63, Professor Cayley considers the theory of the Quinquisection. He gives the expression of the fifteen coefficients in X^2, XY, XZ in terms of five quantities α, β, ϑ, f, k, connected by two quadric equations and by the linear equation

$$\alpha+\beta+\vartheta=\tfrac{1}{5}(p-1).$$

Professor Cayley finds the expression of these coefficients in terms of α, β, ϑ, f, k, by induction from numerical examples, but refers to Mr. F. S. Carey's Trinity Fellowship Dissertation (1884) for proofs of equivalent formulae. From the form of Mr. Carey's results, I imagine that his proof is not the one here given. Using Professor Cayley's notation, h is the number of positive integral solutions (with regard to λ, μ, ν) of the congruence

$$\gamma^{5\lambda}+\gamma^{5\mu+1}\equiv\gamma^{5\nu+2}\,(\text{mod. } p),$$

where γ is the selected prime root, and therefore $\gamma^{\frac{1}{2}(p-1)}\equiv-1\,(\text{mod. } p)$; *i. e.*,

$$\gamma^{5\rho}\equiv-1\,(\text{mod. } p)\ [p=10\rho+1].$$

The congruence is therefore equivalent to

$$\gamma^{5(\rho+\mu)+1}+\gamma^{5\nu+2}\equiv\gamma^{5\lambda},$$

the number of solutions of which gives the coefficient of X in the product YZ; *i. e.*, j, $\therefore h=j$.

This method is applicable to the coefficients in the products of the periods, but not to the coefficients in the squares of the periods, for, since there are five periods, complementary terms come in the same period. Thus the only systems of equalities that we get are

$$h = j = l; \quad i = n = o.$$

Write $\frac{1}{5}(p-1) = \lambda$. We have then (see Prof. Cayley's paper)

$$\left.\begin{array}{c} \Sigma f = \lambda. \qquad \Sigma k = \lambda. \qquad \Sigma a = -1 - 4\lambda. \\ \alpha + \beta + \vartheta = \lambda. \end{array}\right\}. \tag{3}$$

For the equation for Quinquisection we have

$$\text{coefficient of } \eta^4 = -\Sigma X = 1;$$
$$\text{coefficient of } \eta^3 = \Sigma(XY + XZ) = -\Sigma f - \Sigma k = -2\lambda.$$

The products of the periods in threes are such as ZWT or else such as YWT.

$$\begin{aligned} \Sigma ZWT = {} & X.YZ + Y.ZW + Z.WT + W.TX + T.XY \\ = {} & X(j, f, g, h, i) + Y(i, j, f, g, h) + Z(h, i, j, f, g) + W(g, h, j, i, f) \\ & + T(f, g, h, i, j) \\ = {} & \quad f(XY + YZ + ZW + WT + TX) \\ & + g(XZ + YW + ZT + WX + TY) \\ & + h(XW + YT + ZX + WY + TZ) \\ & + i(XT + YX + ZY + WZ + TW) \\ & + j(X^2 + Y^2 + Z^2 + W^2 + T^2) \\ = {} & (f + i)\Sigma f.\Sigma X + (g + h)\Sigma k.\Sigma X + j\Sigma a.\Sigma X \\ = {} & -\lambda(f + g + h + i) + j(1 + 4\lambda) \qquad \text{[by (3)} \\ = {} & -\lambda(f + g + h + i + j) + j(1 + 5\lambda) \\ = {} & -\lambda^2 + pj. \end{aligned}$$

Similarly,

$$\begin{aligned} \Sigma YWT = {} & X.ZW + Y.WT + Z.TX + W.XY + T.YZ \\ = {} & -\lambda(f + g + h + j) + i(1 + 4\lambda) \\ = {} & -\lambda^2 + pi. \end{aligned}$$

$$\begin{aligned} \therefore \text{ coefficient of } \eta^2 = {} & -\Sigma(ZWT + YWT) \\ = {} & 2\lambda^2 - p(i + j) \\ = {} & 2\lambda^2 - p(\alpha + \beta). \end{aligned}$$

Write $\qquad s_1$ for $fg + gh + hi + ij + jf$

and $\qquad s_2$ for $fh + gi + hj + if + jg$,

then
$$\begin{aligned}\Sigma YZWT &= \Sigma(j, f, g, h, i)(h, i, j, f, g)\\ &= s_2\Sigma X^2 + (s_1 + s_2)\Sigma XY + (s_1 + \Sigma f^2)\,\Sigma XZ\\ &= -\Sigma f^2.\Sigma k - s_1(\Sigma f + \Sigma k) - s_2(\Sigma a + \Sigma f)\\ &= -\lambda\Sigma f^2 - 2\lambda s_1 + (1 + 3\lambda)s_2 \qquad \text{[by (3)}\\ &= -\lambda\{\Sigma f^2 + 2s_1 + 2s_2\} + ps_2.\end{aligned}$$

Now
$$\Sigma f^2 + 2s_1 + 2s_2 = [\Sigma f]^2 = \lambda^2, \qquad \text{[by (3)}$$

and
$$\begin{aligned}s_2 &= fh + gi + hj + if + jg\\ &= f\alpha + g\beta + \alpha^2 + \beta f + \alpha g\\ &= (\alpha + \beta)(f + g) + \alpha^2\\ &= (\alpha + \beta)(\vartheta + \alpha) + \alpha^2\\ &= \vartheta(\alpha + \beta) - \alpha\beta,\end{aligned}$$

∴ coefficient of $\eta = -\lambda^3 + p\{\vartheta(\alpha + \beta) - \alpha\beta\}$.

The remaining term I have not succeeded in finding. The equation for Quinquisection is therefore

$$\eta^5 + \eta^4 - 2\frac{p-1}{5}\eta^3 + \left\{2\left(\frac{p-1}{5}\right)^2 - p(\alpha + \beta)\right\}\eta^2 - \left\{\left(\frac{p-1}{5}\right)^3 - p\,[\vartheta(\alpha + \beta) - \alpha\beta]\right\}\eta + [?] = 0,$$

where $\vartheta + \alpha + \beta = \frac{1}{5}(p - 1)$.